MATH 4010 | Tutorial 1 | 15/01/2015

Eg 1: $X = C[0,1]$, $d(x(t), y(t)) = \max_{t\in[0,1]} |x(t) - y(t)|$, $\forall\, x(t), y(t) \in X$.

Show that (X, d) is a complete metric space.

Pf: ① (X, d) is a metric space.

1) It is clear that $d(x,y) \geqslant 0$ and $d(x,y) = 0$ iff $x = y$.

Moreover $d(x,y) = d(y,x)$

2) $\forall\, x, y, z \in X$, $\forall\, t \in [0,1]$

$$|x(t) - y(t)| \leqslant |x(t) - z(t)| + |z(t) - y(t)|$$

Taking the maximum over $[0,1]$, one has

$$d(x,y) \leqslant d(x,z) + d(z,y)$$

② Completeness

Let $\{x_n\} \subset X$ be an arbitrary Cauchy sequence, that is

$\forall\, \varepsilon > 0$, $\exists\, N$ s.t. for $m, n > N$

$$d(x_m, x_n) = \max_{t\in[0,1]} |x_m(t) - x_n(t)| < \varepsilon \qquad (*)$$

Then, $\forall\, t_* \in [0,1]$, $|x_m(t_*) - x_n(t_*)| < \varepsilon$

which implies that $\{x_n(t_*)\}$ is Cauchy sequence in $\mathbb{R}$.

By the completeness of real numbers, $\exists\, x(t_*) \in \mathbb{R}$ s.t.

$$x_n(t_*) \to x(t_*) \text{ as } n \to +\infty.$$

Therefore, $\{x_n\}$ uniformly converge to $x(t)$. So $x(t)$ is continuous.

Now, it remains to show that $d(x_m, x) \to 0$ as $m \to \infty$.

Set $n \to \infty$ in $(*)$, one has

$$\max_{t\in[0,1]} |x_m(t) - x(t)| \leqslant \varepsilon.$$

i.e., $d(x_m, x) \leqslant \varepsilon$

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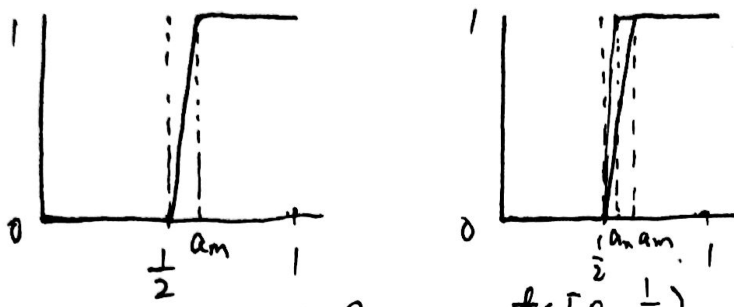
Eg 2. $X = C[0,1]$, $d(x,y) = \int_0^1 |x(t) - y(t)|\, dt$, $\forall x, y \in X$.

Show that (X, d) is not completed.

Pf: It is easy to check (X,d) is a metric space.

Now, we construct a Cauchy sequence as follows:



$$x_n(t) = \begin{cases} 0 & t \in [0, \frac{1}{2}) \\ n(t - \frac{1}{2}) & t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}] \\ 1 & t \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

Then $\forall \varepsilon > 0$,

$$d(x_m, x_n) = \int_0^1 |x_m - x_n|\, dt = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon \quad \text{for } m, n > \frac{1}{\varepsilon}$$

So $\{x_n\}$ is a Cauchy sequence.

Now, $\forall x \in X$,

$$d(x_n, x) = \int_0^1 |x_n(t) - x(t)|\, dt$$
$$= \int_0^{\frac{1}{2}} |x(t)|\, dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |x_n(t) - x(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{n}}^1 |1 - x(t)|\, dt$$

Then $d(x_n, x) \to 0$ as $n \to \infty$ yields that

$$0 = \int_0^{\frac{1}{2}} |x(t)|\, dt = \int_{\frac{1}{2}}^1 |1 - x(t)|\, dt$$

i.e. $x(t) = \begin{cases} 0 & t \in (0, \frac{1}{2}) \\ 1 & t \in (\frac{1}{2}, 1). \end{cases}$

Therefore $x(t)$ is impossible continuous!

That is the Cauchy sequence is not convergent.

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